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A family of Sobolev orthogonal polynomials on the unit ball

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Abstract

A family of orthonormal polynomials on the unit ball B^d of $\mathbb{R}^d$ with respect to the inner product

$$\langle f, g \rangle = \int_{B^d} \Delta[(1-\|x\|^2) f(x)] \Delta[(1-\|x\|^2) g(x)] \, dx,$$

where Δ is the Laplace operator, is constructed explicitly.

MSC: 42A38; 42B08; 42B15

Keywords: Sobolev orthogonal polynomials; Several variables; Unit ball

1. Introduction

In a recent study on the numerical solution of the nonlinear Poisson equation $-\Delta u = f(\cdot, u)$ on the unit disk with zero boundary conditions, Atkinson and Hansen [2] asked the question of finding an explicit orthogonal basis for the inner product defined by

$$\langle f, g \rangle_\Delta := \frac{1}{\pi} \int_{B^2} \Delta[(1-x^2-y^2) f(x, y)] \Delta[(1-x^2-y^2) g(x, y)] \, dx \, dy$$

on the unit disk B^2 of the Euclidean plane, where Δ is the usual Laplace operator. The purpose of this note is to provide an answer for this question.

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We shall consider more generally the analogous inner product on the unit ball B^d in $\mathbb{R}^d$. We call orthogonal polynomials with respect to such an inner product Sobolev orthogonal polynomials. In the theory of orthogonal polynomials of one variable, the name Sobolev is associated with polynomials that are orthogonal with respect to an inner product defined using both functions and their derivatives; see, for example, [4] and the references therein. As far as we know, Sobolev orthogonal polynomials have not been studied in the case of several variables.

Our main result, given in Section 2, is a family of orthonormal polynomials with respect to $\langle\cdot,\cdot\rangle_\Delta$ on B^d that are constructed using spherical harmonics and Jacobi polynomials in Section 2. For $d=1$, orthogonal polynomials with respect to this inner product have been studied recently in [5]. The explicit formula can be used to study further properties of the orthogonal basis. In particular, it turns out that the orthogonal expansion of a function f in this basis can be computed without involving the derivatives of f. This will be discussed in Section 3.

2. Sobolev orthogonal polynomials

For $x \in \mathbb{R}^d$, let $\|x\|$ denote the usual Euclidean norm of x. The unit ball in $\mathbb{R}^d$ is $B^d := \{x : \|x\| \leqslant 1\}$. Its surface is $S^{d-1} := \{x : \|x\| = 1\}$. The volume of B^d and the surface area of S^{d-1} are denoted by $\mathrm{vol}(B^d)$ and ω_{d-1}, respectively,

$$\mathrm{vol}(B^d) = \omega_{d-1}/d \quad \text{and} \quad \omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2).$$

Let $\Pi^d = \mathbb{R}[x_1, \ldots, x_d]$ be the ring of polynomials in d variables and let Π_n^d denote the subspace of polynomials of total degree at most n. We consider the inner product defined on the polynomial space by

$$\langle f, g\rangle_\Delta := \frac{1}{4d^2\,\mathrm{vol}(B^d)} \int_{B^d} \Delta[(1-\|x\|^2)f(x)]\Delta[(1-\|x\|^2)g(x)]\,dx.$$

The constants are chosen so that $\langle 1, 1\rangle_\Delta = 1$. As pointed out in [2], the inner product is well defined and positive definite on Π^d. Let $\mathcal{V}_n^d(\Delta)$ denote the space of orthogonal polynomials of degree n which are orthogonal to all polynomials of lower degree with respect to $\langle f, g\rangle_\Delta$. It follows from the general theory of orthogonal polynomials in several variables [3] that the dimension of $\mathcal{V}_n^d(\Delta)$ is $\binom{n+d-1}{d-1}$. If $\{P_\alpha\}$ is a basis of $\mathcal{V}_n^d(\Delta)$ and $\langle P_\alpha, P_\beta\rangle_\Delta = 0$ whenever $\alpha \neq \beta$, it is called a mutually orthogonal basis. If, in addition, P_α is normalized so that $\langle P_\alpha, P_\alpha\rangle_\Delta = 1$, the basis is called orthonormal. Our objective in this section is to find an explicit orthonormal basis for $\mathcal{V}_n^d(\Delta)$.

The presence of the Laplace operator suggests that we make use of harmonic polynomials, which are homogeneous polynomials that satisfy the equation $\Delta P = 0$. Let $\mathcal{H}_n^d$ denote the space of harmonic polynomials of degree n. It is well known that

$$\dim \mathcal{H}_n^d = \binom{n+d-1}{d-1} - \binom{n+d-3}{d-1} := \sigma_n.$$

The restriction of $Y \in \mathcal{H}_n^d$ on S^{d-1} are called spherical harmonics. They are orthogonal on S^{d-1}. We will use the spherical polar coordinates $x = rx'$ for $x \in \mathbb{R}^d$, $r \geqslant 0$, and $x' \in S^{d-1}$. For $Y \in \mathcal{H}_n^d$ we use the notation $Y(x)$ to denote the harmonic polynomials and use $Y(x')$ to denote the spherical harmonics. This agrees with $x = rx'$ since Y is a homogeneous polynomial, $Y(x) = r^n Y(x')$. Throughout this paper, we use the notation $\{Y_\nu^n : 1 \leqslant \nu \leqslant \sigma_n\}$ to denote an orthonormal basis for $\mathcal{H}_n^d$, that is,

$$\frac{1}{\omega_{d-1}} \int_{S^{d-1}} Y_\mu^n(x') Y_\nu^m(x')\, d\omega(x') = \delta_{\mu,\nu}\delta_{n,m}, \quad 1 \leqslant \mu, \nu \leqslant \sigma_n, \tag{2.1}$$

where $d\omega$ stands for the surface measure on S^{d-1}. In terms of the spherical polar coordinates, $x = rx'$, $r > 0$ and $x' \in S^{d-1}$, the Laplace operator can be written as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0, \tag{2.2}$$

where Δ_0 is the spherical Laplacian on S^{d-1}. It is well-known that

$$\Delta_0 Y(x') = -n(n+d-2)Y(x'), \qquad Y \in \mathcal{H}_n^d, \quad x' \in S^{d-1}. \tag{2.3}$$

The spherical harmonics have been used to construct orthogonal polynomials on the unit ball. For later use, let us mention an orthogonal basis with respect to the inner product

$$\langle f, g\rangle_\mu := c_\mu \int_{B^d} f(x)g(x)W_\mu(x)\, dx, \qquad W_\mu(x) = (1 - \|x\|^2)^\mu,$$

where $\mu > -1$ and c_μ is the normalization constant of W_μ. Let $\mathcal{V}_n^d(W_\mu)$ denote the space of orthogonal polynomials of degree n that are orthogonal to all polynomials of smaller degree based on using $\langle f, g\rangle_\mu$. A mutually orthogonal basis for $\mathcal{V}_n(W_\mu)$ is given by [3]

$$P_{j,\nu}^n(W_\mu; x) = P_j^{(\mu, n-2j+\frac{d-2}{2})}(2\|x\|^2 - 1)Y_\nu^{n-2j}(x), \quad 0 \leqslant j \leqslant n/2, \tag{2.4}$$

where $P_j^{(\alpha,\beta)}$ denotes the Jacobi polynomial of degree j, which is orthogonal with respect to $(1-x)^\alpha(1+x)^\beta$ on $[-1, 1]$, and $\{Y_\nu^{n-2j} : 1 \leqslant j \leqslant \sigma_{n-2j}\}$ is a basis for $\mathcal{H}_{n-2j}^d$.

In view of (2.4) we will look for a basis with respect to $\langle f, g\rangle_\Delta$ in the form of

$$Q_{j,\nu}^n(x) = q_j(2\|x\|^2 - 1)Y_\nu^{n-2j}(x), \qquad 0 \leqslant 2j \leqslant n, \quad Y_\nu^{n-2j} \in \mathcal{H}_{n-2j}^d, \tag{2.5}$$

where q_j is a polynomial of degree j in one variable.

Lemma 2.1. *Let $Q_{j,\nu}^n$ be defined as above. Then*

$$\Delta\left[(1 - \|x\|^2)Q_{j,\nu}^n(x)\right] = 4\left(\mathcal{J}_\beta q_j\right)(2r^2 - 1)Y_\nu^{n-2j}(x),$$

where $\beta = n - 2j + \frac{d-2}{2}$ *and*

$$(\mathcal{J}_\beta q_j)(s) = (1 - s^2)q_j''(s) + (\beta - 1 - (\beta + 3)s)q_j'(s) - (\beta + 1)q_j(s).$$

Proof. Using spherical-polar coordinates, we can use (2.3) for the spherical part of Δ, then the radial part of Δ in (2.2) gives, after a tedious calculation, that

$$\begin{aligned}\Delta\left[(1 - \|x\|^2)Q_{j,\nu}^n(x)\right] &= \Delta\left[(1 - r^2)q_j(2r^2 - 1)r^{n-2j}Y_\nu^{n-2j}(x')\right] \\ &= 4r^{n-2j}\Big[4r^2(1 - r^2)q_j''(2r^2 - 1) \\ &\quad + 2((\beta + 1) - (\beta + 3)r^2)q_j'(2r^2 - 1) \\ &\quad - (\beta + 1)q_j(2r^2 - 1)\Big]Y_\nu^{n-2j}(x').\end{aligned}$$

Setting $s \mapsto 2r^2 - 1$ gives the stated result. □

Lemma 2.2. *Let $p_k^\beta \in \Pi_k := \Pi_k^1$ be orthogonal with respect to the inner product*

$$(f, g)_\beta := \int_{-1}^{1} (\mathcal{J}_\beta f)(s)(\mathcal{J}_\beta g)(s)(1+s)^\beta \, ds, \qquad \beta > -1.$$

Then the polynomials $Q_{j,\nu}^n$ in (2.5) with $q_j = p_j^{\beta_{n-2j}}$, where $\beta_k = k + (d-2)/2$, form a mutually orthogonal basis for $\mathcal{V}_n^d(\Delta)$.

Proof. It is easy to see that $(f, g)_\beta$ is indeed a positive definite inner product on the space of polynomials of one variables, so that the orthogonal polynomials with respect to $(f, g)_\beta$ exist (see Lemma 2.3 below). Using the formula

$$\int_{B^d} f(x)\, dx = \int_0^1 r^{d-1} \int_{S^{d-1}} f(rx')\, d\omega(x')\, dr,$$

the definition of $Q_{j,\nu}^n$ and (2.1) shows immediately that

$$\langle Q_{j,\nu}^n, Q_{j',\nu'}^{n'} \rangle_\Delta := \delta_{\nu,\nu'} \delta_{n-2j,n'-2j'} \times \frac{1}{4d} \int_0^1 r^{d+2(n-2j)-1} 4^2 (\mathcal{J}_{\beta_{n-2j}} q_j)(2r^2-1)(\mathcal{J}_{\beta_{n'-2j'}} q_{j'})(2r^2-1)\, dr.$$

In the nonzero case we have $\beta_{n-2j} = \beta_{n'-2j'}$. Thus, a change of variable $r \mapsto \sqrt{(1+s)/2}$ shows that

$$\langle Q_{j,\nu}^n, Q_{j',\nu'}^{n'} \rangle_\Delta = \delta_{\nu,\nu'} \delta_{n-2j,n'-2j'} \frac{1}{d 2^{\beta_{n-2j}}} (q_j, q_{j'})_{\beta_{n-2j}}, \tag{2.6}$$

which proves the stated result. □

We note that $q_j^{\beta_{n-2j}}$ should be understood as one member (of degree j) in the orthogonal family $\{q_k^{\beta_{n-2j}}\}$.

Lemma 2.3. *The polynomials p_j^β defined by*

$$p_0^\beta(s) = 1, \qquad p_j^\beta(s) = (1-s) P_{j-1}^{(2,\beta)}(s), \quad j \geqslant 1$$

are orthogonal with respect to the inner product $(f, g)_\beta$.

Proof. We need the following property of the Jacobi polynomials [6, p. 71],

$$(1-s) P_{j-1}^{(2,\beta)}(s) = \frac{2}{2j+\beta+1} \left[(j+1) P_{j-1}^{(1,\beta)}(s) - j P_j^{(1,\beta)}(s) \right]. \tag{2.7}$$

The Jacobi polynomial $P_{j-1}^{(1,\beta)}$ satisfies a differential equation

$$(1-s^2) y'' + (-1+\beta-(3+\beta)s) y' + (j-1)(j+\beta+1) y = 0.$$

Using these two facts, we easily deduce that

$$\begin{aligned}\frac{2j+\beta+1}{2}\mathcal{J}_\beta\left[(1-s)P_{j-1}^{(2,\beta)}(s)\right] &= (j+1)\mathcal{J}_\beta P_{j-1}^{(1,\beta)}(s) - j\mathcal{J}_\beta P_j^{(1,\beta)}(s)\\ &= (j+1)\left[(-(j-1)(j+\beta+1)-(\beta+1))P_{j-1}^{(1,\beta)}(s)\right]\\ &\quad -j\left[(-j(j+\beta+2)-(\beta+1))P_j^{(1,\beta)}(s)\right]\\ &= -j(j+1)\left[(j+\beta)P_{j-1}^{(1,\beta)}(s) - (j+\beta+1)P_j^{(1,\beta)}(s)\right].\end{aligned}$$

We need yet another formula of Jacobi polynomials [1, p. 782, (22.7.18)],

$$(2j+\beta+1)P_j^{(0,\beta)}(s) = (j+\beta+1)P_j^{(1,\beta)}(s) - (j+\beta)P_{j-1}^{(1,\beta)}(s) \tag{2.8}$$

which implies immediately that

$$\mathcal{J}_\beta\left[(1-s)P_{j-1}^{(2,\beta)}(s)\right] = 2j(j+1)P_j^{(0,\beta)}(s). \tag{2.9}$$

Hence, for $j, j' \geqslant 1$, we conclude that

$$\begin{aligned}(p_j^\beta, p_{j'}^\beta)_\beta &= \int_{-1}^1 \mathcal{J}_\beta\left[(1-s)P_{j-1}^{(2,\beta)}(s)\right]\mathcal{J}_\beta\left[(1-s)P_{j'-1}^{(2,\beta)}(s)\right](1+s)^\beta)\,ds\\ &= 4j(j+1)j'(j'+1)\int_{-1}^1 P_j^{(0,\beta)}(s)P_{j'}^{(0,\beta)}(s)(1+s)^\beta\,ds = 0\end{aligned} \tag{2.10}$$

whenever $j \neq j'$. Furthermore, for $j \geqslant 1$, we have

$$(p_0^\beta, p_j^\beta)_\beta = -2j(j+1)(\beta+1)\int_{-1}^1 P_j^{(0,\beta)}(s)(1+s)^\beta\,ds = 0$$

since $(\mathcal{J}_\beta p_0^\beta)(s) = (\mathcal{J}_\beta 1) = -(\beta+1)$. □

As a consequence of the above lemmas, we have found a mutually orthogonal basis with respect to $\langle\cdot,\cdot\rangle_\Delta$.

Theorem 2.4. *A mutually orthogonal basis for* $\mathcal{V}_n^d(\Delta)$ *is given by*

$$\begin{aligned}Q_{0,v}^n(x) &= Y_v^n(x),\\ Q_{j,v}^n(x) &= (1-\|x\|^2)P_{j-1}^{(2,n-2j+\frac{d-2}{2})}(2\|x\|^2-1)Y_v^{n-2j}(x), \quad 1 \leqslant j \leqslant \frac{n}{2},\end{aligned} \tag{2.11}$$

where $\{Y_v^{n-2j} : 1 \leqslant v \leqslant \sigma_{n-2j}\}$ *is an orthonormal basis of* $\mathcal{H}_{n-2j}^d$. *Furthermore,*

$$\langle Q_{0,v}^n, Q_{0,v}^n\rangle_\Delta = \frac{2n+d}{d}, \qquad \langle Q_{j,v}^n, Q_{j,v}^n\rangle_\Delta = \frac{8j^2(j+1)^2}{d(n+d/2)}. \tag{2.12}$$

Proof. The fact that $Q_{j,v}^n \in \mathcal{V}_n^d(\Delta)$ follows from Lemmas 2.2 and 2.3. To compute the norm of $Q_{0,v}^n$ we use the fact that

$$\Delta[(1-\|x\|^2)Y_v^{n-2j}(x)] = -2dY_v^n(x) - 4\langle x, \nabla\rangle Y_v^n = -2(d+2n)Y_v^n(x) \tag{2.13}$$

by Euler's formula on homogeneous polynomials, which shows that

$$\langle Q^n_{0,v}, Q^n_{0,v}\rangle_\Delta = \frac{(2n+d)^2}{d}\int_0^1 r^{d-1+2n}\,dr\,\frac{1}{\omega_{d-1}}\int_{S^{d-1}}\left[Y^n_v(x)\right]^2\,dx = \frac{2n+d}{d}.$$

Furthermore, using Eqs. (2.6) and (2.10), we have

$$\begin{aligned}\langle Q^n_{j,v}, Q^n_{j,v}\rangle_\Delta &= \frac{1}{d2^{\beta_j}}(p_j, p_{j'})_{\beta_j} = \frac{4j^2(j+1)^2}{d2^{\beta_j}}\int_{-1}^1\left[P_j^{(0,\beta_j)}(s)\right]^2(1+s)^{\beta_j}\,ds\\ &= \frac{8j^2(j+1)^2}{d(\beta_j+2j+1)} = \frac{8j^2(j+1)^2}{d(n+d/2)},\end{aligned}$$

where we have used the well-known formula for the norm of the Jacobi polynomial (see, for example, [6, p. 68]). □

The explicit formula of the basis (2.11) leads to the following interesting result, which relates $\mathcal{V}^d_n(\Delta)$ to orthogonal polynomials with respect to $W_2(x) = (1-\|x\|)^2$.

Corollary 2.5. *For* $n \geqslant 1$,

$$\mathcal{V}^d_n(\Delta) = \mathcal{H}^d_n \oplus (1-\|x\|^2)\mathcal{V}^d_{n-1}(W_2).$$

Proof. Using the basis (2.4) for $\mathcal{V}^d_{n-1}(W_2)$, it follows that we actually have

$$Q^n_{j,v}(x) = (1-\|x\|^2)P^{n-2}_{j-1,v}(W_2; x) \tag{2.14}$$

for $j \geqslant 1$, from which the stated result follows. □

In the case of $d=2$, an orthonormal basis for the space $\mathcal{H}^2_k$ is given by

$$Y^n_1(x,y) = \sqrt{\tfrac{1}{2}}r^n\cos n\theta \quad\text{and}\quad Y^n_2(x,y) = \sqrt{\tfrac{1}{2}}r^n\sin n\theta$$

in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Hence, a mutually orthogonal basis for $\mathcal{V}^2_n(\Delta)$ is given by

$$\begin{aligned}&Q^n_{0,1}(x,y) = Y^n_1(x,y), \qquad Q^n_{0,2}(x,y) = Y^n_2(x,y),\\ &Q^n_{j,1}(x,y) = (1-x^2-y^2)P^{(2,n-2j)}_{j-1}(2x^2+2y^2-1)Y^{n-2j}_1(x,y), \quad 1\leqslant j\leqslant \frac{n}{2},\\ &Q^n_{j,2}(x,y) = (1-x^2-y^2)P^{(2,n-2j)}_{j-1}(2x^2+2y^2-1)Y^{n-2j}_2(x,y), \quad 1\leqslant j\leqslant \frac{n-1}{2},\end{aligned}$$

which becomes an orthonormal basis upon dividing by the square root of the norm given by (2.12). Without normalization, this gives

$$\begin{aligned}&\mathcal{V}^2_1(\Delta) = \operatorname{span}\{x, y\}, \qquad \mathcal{V}^2_2(\Delta) = \operatorname{span}\{x^2-y^2, xy, 1-x^2-y^2\},\\ &\mathcal{V}^2_3(\Delta) = \operatorname{span}\{x^3-3xy^2, 3y^3-x^2y, x(1-x^2-y^2), y(1-x^2-y^2)\},\end{aligned}$$

for example.

3. Expansions in Sobolev orthogonal polynomials

Let $H^2(B^d)$ denote the space of functions for which $\langle f, f\rangle_\Delta$ is finite. This is not the L^2 space on B^d since the definition of $\langle\cdot,\cdot\rangle_\Delta$ require that f has second-order derivatives. Nevertheless, the standard Hilbert space theory shows that every $f \in H^2(B^d)$ can be expanded into a series in Sobolev orthogonal polynomials. In other words,

$$H^2(B^d) = \sum_{n=0}^{\infty} \oplus \mathcal{V}_n^d(\Delta): \qquad f = \sum_{n=0}^{\infty} \operatorname{proj}_n f,$$

where $\operatorname{proj}_n : H^2(B^d) \mapsto \mathcal{V}_n^d(\Delta)$ is the projection operator, which can be written in terms of the orthonormal basis (2.11) as

$$\operatorname{proj}_n f(x) = \sum_{0\leqslant j\leqslant n/2} H_j^{-1} \sum_{v=0}^{\sigma_{n-2j}} \widehat{f}^n_{j,v} Q^n_{j,v}(x), \qquad \widehat{f}^n_{j,v} = \langle f, Q^n_{j,v}\rangle_\Delta, \tag{3.1}$$

where $H_j = \langle Q^n_{j,v}, Q^n_{j,v}\rangle_\Delta$ are independent of v as shown in (2.12). Let $P_n^\Delta(x, y)$ denote the reproducing kernel of $\mathcal{V}_n^d(\Delta)$. In terms of the orthonormal basis (2.11) in the previous section, the reproducing kernel can be written as

$$P_n^\Delta(x, y) = \sum_{0\leqslant j\leqslant n/2} H_j^{-1} \sum_v Q^n_{j,v}(x) Q^n_{j,v}(y).$$

The projection operator can be written as an integral operator with P_n^Δ as its kernel, which means that

$$\begin{aligned}\operatorname{proj}_n f(x) &= \langle f, P_n^\Delta(x,\cdot)\rangle_\Delta \\ &= \frac{1}{4d^2 \operatorname{vol}(B^d)} \int_{B^d} \Delta[(1-\|y\|^2) f(y)] \Delta[(1-\|y\|^2) P_n^\Delta(x, y)]\, dy,\end{aligned}$$

where Δ is applied on y variable.

It turns out that the orthogonal expansion can be computed without involving derivatives of f.

Theorem 3.1. *For $j \geqslant 1$, let $\beta_j = n - 2j + (d-2)/2$; then*

$$\begin{aligned}\widehat{f}^n_{j,v} = \frac{8j(j+1)}{d^2 \operatorname{vol}(B^d)} \Bigg[&(\beta_j + j)(\beta_j + j + 1) \int_{B^d} f(x) Q^n_{j,v}(x)\, dx \\ &- \frac{1}{2} \int_{S^{d-1}} f(y') Y_v^{n-2j}(y')\, d\omega(y') \Bigg];\end{aligned} \tag{3.2}$$

furthermore, for $j = 0$,

$$\widehat{f}^n_{0,v} = \frac{d+2n}{d} \frac{1}{\omega_d} \int_{S^{d-1}} Y_v^{n-2j}(y') f(y')\, d\omega(y').$$

Proof. By (2.4), $P^n_{j,v}(W_0; x) = P_j^{(0,\beta_j)}(2\|x\|^2 - 1) Y_v^{n-2j}(x)$. Let $j \geqslant 1$. By Lemma 2.1 and (2.9),

$$\Delta\left[(1-\|x\|^2) Q^n_{j,v}(x)\right] = 8j(j+1) P^n_{j,v}(W_0; x).$$

Applying Green's identity

$$\int_{B^d} (u\Delta v - v\Delta u)\, dx = \int_{S^{d-1}} \left(\frac{\partial v}{\partial n} u - \frac{\partial u}{\partial n} v \right) d\omega$$

with $v(x) = (1 - \|x\|^2) f(x)$ and $u = Q^n_{j\beta}$ shows then

$$\begin{aligned} \widehat{f}^n_{j,v} &= \frac{8j(j+1)}{4d^2 \operatorname{vol}(B^d)} \int_{B^d} \Delta \left[(1-\|x\|^2) f(x) \right] P^n_{j,v}(W_0; x)\, dx \\ &= \frac{2j(j+1)}{d^2 \operatorname{vol}(B^d)} \left[\int_{B^d} (1-\|x\|^2) f(x) \Delta P^n_{j,v}(W_0; x)\, dx \right. \\ &\quad \left. - 2 \int_{S^{d-1}} Y^{n-2j}_v (x') f(x')\, d\omega \right], \end{aligned} \tag{3.3}$$

where we have used the fact that $P^{(0,\beta)}_j(1) = 1$. Let $\partial P^{(0,\beta)}$ denote the derivative of $P^{(0,\beta)}$. Using (2.2) and (2.3), it is easy to see that

$$\begin{aligned} \Delta[P^n_{j,v}(W_0; x)] = 8 \Big[& 2r^2 \partial^2 P^{(0,\beta_j)}_j (2r^2 - 1) \\ & + (n - 2j + d/2) \partial P^{(0,\beta_j)}_j (2r^2 - 1) \Big] Y^{n-2j}_v (x). \end{aligned}$$

Let us denote the expression in the square bracket by M_j. The Jacobi polynomial $P^{(0,\beta)}_j(s)$ satisfies the differential equation

$$(1 - s^2) y'' - (-\beta + (\beta+2)s) y' + j(j+\beta+1) y = 0.$$

Hence, changing variable $2r^2 - 1 \mapsto s$, we conclude that

$$2(1-r^2) M_j = -j(j+\beta_j+1) P^{(0,\beta_j)}_j (s) + \tfrac{1}{2}(j+\beta_j+1)(1+s) P^{(1,\beta_j+1)}_{j-1}(s).$$

On the other hand, using (2.8), (2.7), [6, (4.5.5)], and the fact that [1, p. 782]

$$(2j+\beta+1)(1+s) P^{(1,\beta+1)}_{j-1}(s) = 2(j+\beta) P^{(1,\beta)}_j (s) + 2j P^{(1,\beta)}_{j-1}(s)$$

we conclude that

$$\begin{aligned} 2(1-r^2) M_j &= \frac{(\beta_j + j + 1)(\beta_j + j)}{2j + \beta_j + 1} \left[-j P^{(1,\beta_j)}_{j-1}(s) + (j+1) P^{(1,\beta_j)}_j (s) \right] \\ &= \frac{1}{2} (\beta_j + j + 1)(\beta_j + j)(1 - s) P^{(2,\beta_j)}_{j-1}(s) \\ &= (\beta_j + j + 1)(\beta_j + j)(1 - r^2) P^{(2,\beta_j)}_{j-1}(2r^2 - 1). \end{aligned}$$

Consequently, we have proved that

$$\begin{aligned} (1 - \|x\|^2) \Delta[P^n_{j,v}(W_0; x)] &= 4(\beta_j + j + 1)(\beta_j + j)(1 - r^2) P^{(2,\beta_j)}_{j-1}(2r^2 - 1) Y^{n-2j}_v (x) \\ &= 4(\beta_j + j + 1)(\beta_j + j) Q^n_{j,v}(x) \end{aligned}$$

which leads to the stated result for $j \geqslant 1$ by (3.3). The proof of $j = 0$ is similar but easier, in which we need to use (2.13). $\square$

Let us denote by $P_n(W_\mu; x, y)$ the reproducing kernel of $\mathcal{V}_n^d(W_\mu)$, which can be written as

$$P_n(W_\mu; x, y) = \sum_{|\alpha|=n} A_{\alpha,\mu}^{-1} P_\alpha(W_\mu; x) P_\alpha(W_\mu; y),$$

where $A_{\alpha,\mu} = c_\mu \int_{B^d} [P_\alpha(W_\mu; y)]^2 W_\mu(y)\, dy$ in which c_μ is the normalization of W_μ. Let us also denote by $C_n^\lambda(t)$ the Gegenbauer polynomial of degree n, and by $x \cdot y$ the usual dot product of $x, y \in \mathbb{R}^d$.

Corollary 3.2. *For $f \in H^2(B^d)$ and $x \in B^d$,*

$$\operatorname{proj}_n f(x) = Y_n f(x) + (1 - \|x\|^2) \frac{4}{\binom{d}{2} \operatorname{vol}(B^d)} \int_{B^d} f(y) P_{n-2}(W_2; x, y)(1 - \|y\|^2)\, dy$$
$$- \frac{(n + d/2)}{4} (1 - \|x\|^2) \sum_{1 \leqslant j \leqslant n/2} \frac{P_{j-1}^{(2, n-2j+\frac{d-2}{2})}(2\|x\|^2 - 1)}{P_{j-1}^{(2, n-2j+\frac{d-2}{2})}(1)} Y_{n-2j} f(x),$$

where with $x' = x/\|x\| \in S^{d-1}$,

$$Y_m f(x) = \|x\|^m \int_{S^{d-1}} f(y') \frac{m + (d-2)/2}{(d-2)/2} C_m^{\frac{d-2}{2}}(x \cdot y')\, d\omega(y').$$

Proof. The values of $H_j = \langle Q_{j,v}^n, Q_{j,v}^n \rangle_\Delta$ are given in (2.12). It follows immediately that

$$\sum_{v=1}^{\sigma_n} H_0^{-1} \widehat{f}_{0,v}^n Q_{0,v}^n(x) = \frac{1}{\omega_{n-1}} \int_{S^{d-1}} f(y') \sum_{v=1}^{\sigma_n} Y_v^n(y') Y_v^n(x)\, d\omega(y') = Y_n f(x),$$

where the last step follows from the summation formula of spherical harmonics,

$$\sum_{v=1}^{\sigma_n} Y_v^n(x) Y_v^n(y) = \|x\|^n \sum_{v=1}^{\sigma_n} Y_v^n(x') Y_v^n(y) = \|x\|^n \frac{n + (d-2)/2}{(d-2)/2} C_n^{\frac{d-2}{2}}(x' \cdot y)$$

for $x', y \in S^{d-1}$. Furthermore, setting $f = Q_{j,v}^n$ with $j \geqslant 1$ in (3.2) also shows

$$H_j = \frac{8j(j+1)}{d^2 \operatorname{vol}(B^d)} (\beta_j + j)(\beta_j + j + 1) \int_{B^d} [Q_{j,v}^n(x)]^2\, dx.$$

Hence, it follows from (3.2) and (2.12) that

$$H_j^{-1} \widehat{f}_{j,v}^n = \frac{\int_{B^d} f(y) Q_{j,v}^n(y)\, dy}{\int_{B^d} [Q_{j,v}^n(y)]^2\, dy} - \frac{n + d/2}{2j(j+1)} \frac{1}{\omega_{d-1}} \int_{S^{d-1}} f(y') Y_v^{n-2j}(y')\, d\omega(y'). \quad (3.4)$$

The relation (2.14) readily shows that

$$\int_{B^d} [Q_{j,v}^n(y)]^2\, dy = \frac{1}{4} \binom{d}{2} \operatorname{vol}(B^d) c_2 \int_{B^d} [P_{j,v}^n(W_2; y)]^2 W_2(y)\, dy. \quad (3.5)$$

We multiply (3.4) by $Q_{j,v}^n(x)$ and sum over v and j. Using (3.5) and the fact that $P_{j-1}^{(2, n-2j+\frac{d-2}{2})}(1) = j(j+1)/2$, the stated result follows from (3.2) and (3.1). □

It follows from this corollary that the orthogonal expansion of f with respect to $\langle \cdot, \cdot \rangle_\Delta$ coincides with the spherical harmonic expansion of f when restricted on S^{d-1}.

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